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Characterization of Small Solutions in Functional Differential Equations

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Abstract—A characterization is given of the initial functions which yield small solutions in a class of linear, autonomous, retarded functional differential equations.

Keywords—Retarded functional differential equation, Partial multiplicities, Jordan form, Generalized left characteristic matrix equation, Transformation, Small solution.

1. INTRODUCTION

Consider the delay system $\mathcal{S}_d$ described by the linear autonomous retarded functional differential equation (rfde)

$$\mathcal{S}_d : \dot{x}(t) = \int_{-r}^{0} d\alpha(\theta)x(t+\theta), \tag{1.1}$$

where $t \in (0, \infty)$ denotes time, $r \in (0, \infty)$ denotes the system memory span, the instantaneous state $x(t) \in \mathbb{R}^n$ and $\alpha \in \mathrm{BV}([-r, 0]; \mathbb{R}^{n\times n})$, the class of $n \times n$ matrix valued functions of bounded variation on $[-r, 0]$. The state of $\mathcal{S}_d$ is the segment function x_t, defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$. In this notation, the initial function of (1.1) can be written as

$$x_0(\theta) = \phi(\theta) - r \leq \theta \leq 0, \tag{1.2}$$

where $\phi \in \mathrm{C}([-r, 0]; \mathbb{R}^n)$, the class of $\mathbb{R}^n$-valued continuous function on $[-r, 0]$. By the solution of $\mathcal{S}_d$ is meant the continuous function $x \in \mathrm{C}([-r, \infty); \mathbb{R}^n)$ for which $x_0(\theta) = \phi(\theta)$ and which satisfies (1.1) for $t > 0$.

The linear *nonautonomous* rfde $\dot{x}(t) = -2te^{1-2t}x(t-1)$ admits the solution $x(t) = e^{-t^2}$ on $[-1, \infty)$. See [1, p. 97]. This solution approaches zero faster than any exponential, but is never zero. Such a solution is said to be small. More precisely, corresponding to an initial function $\phi \neq 0$, the solution of an rfde is said to be a small solution if $\lim_{t\to\infty} e^{kt}x(t) = 0 \,\forall\, k \in \mathbb{R}$. The question then arises as to whether a linear *autonomous* rfde $\mathcal{S}_d$ may admit a small solution [1,2]. Further research [1] has shown that if $x(\cdot)$ is a small solution of $\mathcal{S}_d$, then $x(t) = 0$ for $t \geq nr - E\det\Delta(\lambda))$ where $\Delta(\lambda) = \lambda I_n - \int_{-r}^{0} e^{\lambda\theta} d\alpha(\theta)$ is the system characteristic matrix and

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$E(\det \Delta(\lambda)) = \overline{\lim}_{|\lambda|\to\infty}(\ln|\det \Delta(\lambda)|)/|\lambda|$ is the exponential type of $\det \Delta(\lambda)$. See also [3,4] and references therein for further background on the subject.

In [1], a small solution is characterized through analysis of its exponential type. Our aim is to obtain a spectral characterization based on a transformation [5,6] of S_d to an ordinary differential equation. Following [6], the essential aspects of the transformation theory are summarized below.

2. BACKGROUND MATERIAL

2.1. Jordan Form

Let $T(t) : C([-r,0];\mathbb{R}^n) \to C([-r,0];\mathbb{R}^n)$ defined by $x_t = T(t)\phi$ denote the solution operator of (1.1). Following [1], $\{T(t); t \geq 0\}$ is a C_0 semigroup of bounded linear operators whose infinitesimal generator $\mathcal{A}$ has a countable point spectrum given by $\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \det \Delta(\lambda) = 0\}$ and each eigenvalue has a finite algebraic multiplicity. Let $\mathbb{C}^+_{-\nu_0} = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq -\nu_0,\ \nu_0 \geq 0\}$ denote an arbitrary closed right half of the complex plane bounded by $\lambda = -\nu_0$. It is known [1], that for any $\nu_0 \geq 0$, there are finitely many eigenvalues in $\mathbb{C}^+_{-\nu_0}$. Let λ_j, $j = 1,2,\dots,N$, be the distinct eigenvalues in $\mathbb{C}^+_{-\nu_0}$. For each distinct eigenvalue λ_j with algebraic multiplicity m_j, we define a Jordan cell $J_j(\lambda_j) \in \mathbb{C}^{m_j\times m_j}$ by $J_j = \bigoplus_{l=1}^{g_j} J_j^l(\lambda_j)$, where $J_j^l \in \mathbb{C}^{m_j^l\times m_j^l}$. Thus, $m_j = \sum_{l=1}^{g_j} m_j^l$ where g_j represents the number of subcells in J_j. Extending this to all the N distinct eigenvalues in $\mathbb{C}^+_{-\nu_0}$ yields the Jordan matrix

$$J = \bigoplus_{j=1}^{N} J_j = \bigoplus_{j=1}^{N}\bigoplus_{l=1}^{g_j} J_j^l(\lambda_j), \tag{2.1}$$

where $\sigma(J) = \sigma(\mathcal{A}) \cap \mathbb{C}^+_{-\nu_0}$ and $M = \sum_{j=1}^N m_j$. Knowledge of the number and orders of the subcells in each cell is required to construct the above Jordan matrix $J \in \mathbb{C}^{M\times M}$. This information can be displayed as a Segre characteristic [7, p. 241], namely,

$$\left\{\left(m_1^1, m_1^2, \dots, m_1^{g_1}\right), \left(m_2^1, m_2^2, \dots, m_2^{g_2}\right), \dots, \left(m_N^1, m_N^2, \dots, m_N^{g_N}\right)\right\}.$$

The Segre characteristic can be obtained with the help of the following theorem.

THEOREM 2.1. *Let $\lambda_j \in \sigma(\mathcal{A})$ be an eigenvalue of algebraic multiplicity m_j. Then, the characteristic matrix $\Delta(\lambda)$ admits the representation*

$$\Delta(\lambda) = E(\lambda)\,\mathrm{diag}\left[(\lambda-\lambda_j)^{\kappa_j^1}, (\lambda-\lambda_j)^{\kappa_j^2}, \dots, (\lambda-\lambda_j)^{\kappa_j^n}\right] F(\lambda), \tag{2.2}$$

where $E(\lambda)$ and $F(\lambda)$ are $n\times n$ matrix functions which are analytic and invertible in a neighborhood of λ_j. Furthermore, $\kappa_j^1 \geq \kappa_j^2 \cdots \geq \kappa_j^n \geq 0$ are integers (called partial multiplicities) which are uniquely determined by $\Delta(\lambda)$ and satisfy the relation

$$\kappa_j^1 + \kappa_j^2 + \cdots + \kappa_j^n = m_j. \tag{2.3}$$

This theorem is based on Theorem 1.1.2 and Corollary 1.1.7 of [8] which also provides an algorithm for determining the partial multiplicities. As in [6], we identify the Segre characteristic with the partial multiplicities by setting

$$m_j^l = \kappa_j^l, \qquad l = 1,2,\dots,g_j,$$

where g_j represents the number of nonzero partial multiplicities in the representation (2.2) of $\Delta(\lambda)$. Once the the Jordan matrix $J \in \mathbb{C}^{M\times M}$ is known, the corresponding (left) Jordan chains can be organized through the generalized left characteristic matrix equation (glcme) [5,6]. See a related development in [9].

2.2. Left Jordan Chains

THEOREM 2.2. *Given $\mathcal{S}_d$ and an arbitrary $\nu_0 \geq 0$, there exists a nontrivial $Q \in \mathbb{C}^{M\times n}$ satisfying the glcme, namely,*

$$JQ = \int_{-r}^{0} e^{J\theta} Q \, d\alpha(\theta), \tag{2.4}$$

where $J \in \mathbb{C}^{M\times M}$ is a Jordan matrix such that $\sigma(J) = \sigma(\mathcal{A}) \cap \mathbb{C}^{+}_{-\nu_0}$. More specifically, $0 \neq Q \in \mathbb{C}^{M\times n}$ satisfies (2.4) $\Leftrightarrow$ $co(Q^l_j) \in \ker U'_\alpha(\lambda_j; m^l_j)$. That is,

$$\left[co\left(Q^l_j\right)\right]' U_\alpha\left(\lambda_j; m^l_j\right) = 0, \qquad j = 1,2,\dots,N, \quad l = 1,2,\dots,g_j, \tag{2.5}$$

where $'$ denotes transposition,

$$U_\alpha\left(\lambda_j; m^l_j\right) = \left.\begin{pmatrix} \Delta(\lambda) & \frac{1}{1!}\frac{d\Delta}{d\lambda} & \frac{1}{2!}\frac{d^2\Delta}{d\lambda^2} & \cdots & \frac{1}{(m^l_j-1)!}\frac{d^{m^l_j-1}\Delta}{d\lambda^{m^l_j-1}} \\ 0 & \Delta(\lambda) & \frac{1}{1!}\frac{d\Delta}{d\lambda} & \cdots & \frac{1}{(m^l_j-2)!}\frac{d^{m^l_j-2}\Delta}{d\lambda^{m^l_j-2}} \\ 0 & 0 & \Delta(\lambda) & \cdots & \frac{1}{(m^l_j-3)!}\frac{d^{m^l_j-3}\Delta}{d\lambda^{m^l_j-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Delta(\lambda) \end{pmatrix}\right|_{\lambda=\lambda_j}, \tag{2.6}$$

$$Q = \operatorname{col}\left(Q_j,\ j = 1,2,\dots,N;\ Q_j \in \mathbb{C}^{m_j\times n}\right), \tag{2.7}$$

$$Q_j = \operatorname{col}\left(Q^l_j,\ l = 1,2,\dots,g_j;\ Q^l_j \in \mathbb{C}^{m^l_j\times n}\right), \tag{2.8}$$

and $co : \mathbb{C}^{p\times q} \longrightarrow \mathbb{C}^{1\times pq}$ denotes the isomorphism which carries a $p \times q$ matrix into a pq column vector by stringing the rows of the matrix one after another and then transposing.

See [6] for proof. From (2.5), it is evident that the rows of $Q \in \mathbb{C}^{M\times n}$ are either left eigenvectors or generalized left eigenvectors of $\Delta(\lambda)$. See [6,8] for details. Thus, $Q \neq 0$. As $\nu_0 \to \infty$, $M \to \infty$, but the maximum rank attainable by Q is n. In any event, regardless of whether rank $Q = n$ or not, the pair $(J, Q) \in \mathbb{C}^{M\times M} \times \mathbb{C}^{M\times n}$ satisfying the glcme permits the reduction of $\mathcal{S}_d$ to an ordinary differential equation as follows.

THEOREM 2.3. *Let*

$$z(t) = Qx(t) + \int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)} Q \, d\alpha(\theta) x(\tau)\, d\tau, \qquad t \geq 0, \tag{2.9}$$

where $x(\cdot)$ is any solution of $\mathcal{S}_d$ and the pair $(J, Q) \in \mathbb{C}^{M\times M} \times \mathbb{C}^{M\times n}$ satisfies the glcme (2.4). Then $z(\cdot)$ satisfies the ordinary differential equation

$$\mathcal{S}_0 : \dot{z}(t) = Jz(t), \qquad t \geq 0. \tag{2.10}$$

PROOF. Follows by direct differentiation.

3. MAIN RESULT

THEOREM 3.1. *Given $\mathcal{S}_d$ and $0 \leq \nu_0 < \infty$, let $J \in \mathbb{C}^{M\times M}$ be a Jordan matrix such that $\sigma(J) = \sigma(\mathcal{A}) \cap \mathbb{C}^{+}_{-\nu_0}$ and whose Segre characteristic is determined by the nonzero partial multiplicities of $\Delta(\lambda)$ as above. Let $Q \in \mathbb{C}^{M\times n}$ be a nontrivial solution of the glcme (2.4). Assume that for* *the given ν_0, rank $Q = n$. Then, S_d has a small solution if and only if there exists a nonzero $\phi \in C([-r,0];\mathbb{R}^n)$ such that*

$$Q\phi(0) + \int_{-r}^{0}\int_{\theta}^{0} e^{J(\theta-\tau)} Q\, d\alpha(\theta)\phi(\tau)\, d\tau = 0. \tag{3.1}$$

Such a small solution vanishes no later than $t = r$.

PROOF. $\Rightarrow$ Let $x(\cdot)$ be a small solution of S_d. Then by [1, p. 76], $x(t) = 0$ for $t \geq T_H$ where $T_H = nr - E(\det \Delta(\lambda))$. Inspection of the integral term on the RHS of (2.9) shows it to vanish for $t \geq T_H + r$. Therefore, $z(t) = e^{Jt}z(0) = 0$ for $t \geq T_H + r$ or $z(0) = 0$ as required.
$\Leftarrow$ Suppose that there exists a nontrivial $\phi \in C([-r,0];\mathbb{R}^n)$ such that (3.1) holds; that is, $z(0) = 0$. From (2.10), $z(t) = e^{Jt}z(0) = 0\ \forall\, t \geq 0$. Then, the transformation (2.9), gives

$$Qx(t) = -\int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)} Q\, d\alpha(\theta)x(\tau)\, d\tau, \qquad \forall\, t \geq 0. \tag{3.2}$$

Note that $t - r \leq t + \theta \leq \tau \leq t$, so that for $t \geq r$, $\tau \geq 0$ and the RHS of (3.2) does not explicitly contain the initial function ϕ. Let $y(t) = Qx(t)$. Since rank $Q = n$, $\exists Q_L \in \mathbb{C}^{n\times M}$ such that $Q_L Q = I_n$. Then, for $t \geq r$, (3.2) can be written as

$$y(t) = -\int_{-r}^{0}\int_{t+\theta}^{t} e^{J(t+\theta-\tau)} Q\, d\alpha(\theta) Q_L\, y(\tau)\, d\tau. \tag{3.3}$$

Let $\bigvee_{-r}^{0} \alpha_{lp}$ denote the variation of α_{lp} on $[-r,0]$ and $\|y(t)\| = \max_i |y_i(t)|$. Recall that for $f \in C([a,b])$ and $g \in BV([a,b])$, $|\int_a^b f(t)dg(t)| \leq \sup_{[a,b]} |f(t)| \bigvee_a^b g$. Using these facts and (3.3), we obtain

$$\|y(t)\| \leq \int_{t-r}^{t} k_0 \left\|e^{J(t-\tau)}\right\| \|y(\tau)\|\ d\tau,$$

where k_0 is some constant. Since $t - r \leq \tau \leq t$ or $0 \leq t - \tau \leq r$, it follows that $\|e^{J(t-\tau)}\| \leq e^{\|J\|r}$. Thus, for $t \geq r$, we have

$$\|y(t)\| \leq \int_{0}^{t} k_0 e^{\|J\|r}\, \|y(\tau)\|\ d\tau, \tag{3.4}$$

and by Gronwall's inequality, $\|y(t)\| = \|Qx(t)\| = 0$. That is, for $t \geq r$ and any solution $x(\cdot)$, we have $0 \leq \|x(t)\| = \|Q_L Q x(t)\| \leq \|Q_L\|\ \|Qx(t)\| = 0$ or $x(t) = 0\ \forall\, t \geq r$, whence $\lim_{t\to\infty} e^{kt} x(t) = 0$ for any $k \in \mathbb{R}$. That is, $x(\cdot)$ is a small solution and vanishes no later than $t = r$. This concludes the proof of the theorem.

DISCUSSION.

(i) It would be interesting to obtain a direct condition on $\alpha(\cdot)$ and r to delineate those systems for which the assumption rank $Q = n$ is true.

(ii) It is clear from the foregoing that the question of small solutions reduces to the question of solvability of (3.1) which may be viewed as a system of linear equations for $\phi(0)$.

Applying the Fredholm Alternative Theorem (FAT), it is seen that (3.1) is solvable if and only if

$$\int_{-r}^{0}\int_{\theta}^{0} v e^{J(\theta-\tau)} Q\, d\alpha(\theta)\phi(\tau)\, d\tau = 0,$$

$\forall v' \in \ker(Q')$. Further analysis shows that the above FAT condition is not always upheld, so that (3.1) is, in general, not solvable except under special conditions. Such conditions include the following.

(i) $\sigma(\mathcal{A})$ is finite and rank $Q = n$. In this case, Q^{-1} exists and (3.1) is solvable for $\phi(0)$.

(ii) $\alpha \in \mathrm{BV}([-r,0];\mathbb{R}^{n\times n})$ and $0 \neq \phi \in \mathrm{C}([-r,0];\mathbb{R}^n)$ are such that the integral in (3.1) is identically zero. In that case, the choice of $\phi(0) = 0$ is enough to satisfy (3.1).

Both techniques are illustrated in the following example.

EXAMPLE. Let $S_d : \dot{x}(t) = A_0 x(t) + A_1 x(t-1)$ where $A_0 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Show that this system has small solutions and describe the set of initial functions which yields small solutions.

SOLUTION. The characteristic matrix $\Delta(\lambda) = \begin{pmatrix} \lambda & -e^{-\lambda} & 1 \\ 0 & \lambda & 1-e^{-\lambda} \\ 0 & 0 & \lambda+1 \end{pmatrix}$, $\det \Delta(\lambda) = \lambda^2(\lambda+1)$, so that $\sigma(\mathcal{A}) = \{-1,0,0\} = \{\lambda_1, \lambda_2, \lambda_2\}$ is finite. Therefore, for any $\nu_0 > 1$, $\sigma(\mathcal{A}) \subset \mathbb{C}^{+}_{-\nu_0}$. Here, $N = 2$, $m_1 = 1$, $m_2 = 2$, and $M = 3$. To construct J requires knowledge of the Segre characteristic. For λ_1, $m_1 = 1$, so that, trivially, $m_1^1 = 1$. For λ_2, we employ the algorithm described in the proof of Theorem 1.1.2 of [8], to put $\Delta(\lambda)$ in the *local Smith form* as $\Delta(\lambda) = E(\lambda)\,\mathrm{diag}(\lambda^2, 1, 1) F(\lambda)$ where $E(\lambda)$ and $F(\lambda)$ are $n \times n$ matrix functions invertible in some neighborhood of the origin of $\mathbb{C}$. This shows that $g_2 = 1$ and $m_2^1 (= \kappa_2^1) = 2$, so that the Segre characteristic is $\{(m_1^1), (m_2^1)\}$ $= \{(1),(2)\}$.

THE JORDAN MATRIX. $J_1^1 = (-1)$, $J_2^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow J = J_1 \oplus J_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

LEFT JORDAN CHAINS. For $m_1^1 = 1$, the Jordan chain reduces to a left eigenvector $Q_1^1 = (q_{11}, q_{12}, q_{13})$ given by (2.5) as $Q_1^1 \Delta(\lambda_1) = 0$ to give $Q_1 = Q_1^1 = (0, 0, q_{13})$, $q_{13} \neq 0$. For $m_2^1 = 2$, we have a (left) Jordan chain of order 2 given by

$$(co\, Q_2^1)' \begin{pmatrix} \Delta(\lambda) & \dfrac{d\Delta}{d\lambda}(\lambda) \\ 0 & \Delta(\lambda) \end{pmatrix}_{\lambda=0} = 0.$$

Letting $(co\, Q_2^1)' = (q_{21}, q_{22}, q_{23}, q_{31}, q_{32}, q_{33})$ yields $q_{21} = 0$, $q_{23} = 0$, $q_{31} = q_{22}$, $2q_{22} + q_{33} = 0$, whence $Q_2 = Q_2^1 = \begin{pmatrix} q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = \begin{pmatrix} 0 & q_{22} & 0 \\ q_{22} & q_{32} & -2q_{22} \end{pmatrix}$. Note that for a chain of order 2, $q_{22} \neq 0$. Therefore,

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q_{13} \\ 0 & q_{22} & 0 \\ q_{22} & q_{32} & -2q_{22} \end{pmatrix}$$

and rank $Q = 3 = n$. With the above computations, it is verified that the glcme is satisfied. Since the spectrum is finite and rank $Q = n$, this system has small solutions. From (3.1), the set of initial functions which yield small solutions is given by $\phi \in \mathrm{C}([-r,0];\mathbb{R}^n)$ such that

$$\phi(0) = -\int_{-r}^{0} \int_{\theta}^{0} Q^{-1} e^{J(\theta-\tau)} Q \, d\alpha(\theta) \phi(\tau)\, d\tau = -\int_{-r}^{0} Q^{-1} e^{-J(r+\tau)} Q A_1 \phi(\tau)\, d\tau. \tag{3.5}$$

This gives

$$\phi_3(0) = 0, \qquad \phi_2(0) = -\int_{-1}^{0} \phi_3(\tau)\, d\tau, \qquad \phi_1(0) = \int_{-1}^{0} [(1+\tau)\phi_3(\tau) - \phi_2(\tau)]\, d\tau.$$

Alternatively, choosing $\phi(\tau) = [\phi_1(\tau), 0, 0]$ yields $A_1 \phi(\tau) = 0$, so that the RHS of (3.5) is identically zero. The satisfaction of (3.1) is then guaranteed by putting $\phi(0) = 0$.

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